Proceedings of the
Annual Stability Conference
Structural Stability Research Council
Charlotte, North Carolina, April 11-14, 2023

Rayleigh-Ritz Formulation for Moment Modification Factors on Lateral-Torsional Buckling of I-Beams

Namita Nayak [1], P M Anilkumar[2], Lakshmi Subramanian[3]

Abstract

Lateral-torsional buckling is a major stability concern for long span flexural members, especially during construction and in the negative moment regions of composite I-girders. The classical solution for lateral-torsional buckling was formulated several decades ago for a simply-supported doubly-symmetric I-beam subjected to uniform moment. The classical solution was subsequently modified in order to design for other boundary conditions and loading scenarios, by the use of empirical equations for moment magnification factors. Finite element modelling to calculate lateral-torsional buckling capacities, though accurate, is not practical to implement in every design scenario. The use of empirical equations is also problematic, given that there are different equations for moment modification factors in literature and international design specifications for identical loading and boundary conditions. This paper shows that each of these equations is most accurate within a limited range of applicability. The authors derive the critical lateral-torsional buckling moment solutions using the Rayleigh-Ritz method for beams with both simply-supported and fixed boundary conditions, and subjected to different loading scenarios. Finally, these analytical solutions are shown to compare well with finite element models for a few sample cross-sections.

Keywords: Elastic lateral-torsional buckling, Moment gradient, Rayleigh-Ritz method, End restraints

1. Introduction

Lateral-torsional buckling (LTB) generally occurs when the compression flange is free to move laterally and the cross-section is free to twist. Timoshenko and Gere (1916) derived the elastic lateral-torsional buckling solution for flexurally and torsionally simply-supported doubly-symmetric I-beams subjected to a uniform moment as given in Eq. 1. This is the basic critical moment (M_{ocr}).

$$M_{ocr} = \sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{1}$$

[1] Research Scholar, Indian Institute of Technology Madras, <namitanayak519@gmail.com>
[2] Post Doctoral Scholar, Indian Institute of Technology Madras, <ce18d755@smail.iitm.ac.in>
[3] Assistant Professor, Indian Institute of Technology Madras, <lakshmipriya@iitm.ac.in >

where; E is the Young's modulus of elasticity, I_y is the minor axis moment of inertia, I_w is the warping constant, G is the shear modulus, J is the St.-Venant torsion constant, L is the unbraced length.

Considering that the critical elastic buckling capacity of laterally unsupported beams are influenced by the support and loading conditions, the point of application of load in a cross-section, etc., several methods have been developed in the literature (Kirby and Nethercot 1979; Salvadori 1956; Serna et al. 2006; Wong and Driver 2010) to evaluate the critical moment of laterally unsupported beams for limited support conditions under the action of in-plane bending moments. Design codes and standards of various countries recommend different empirical methods to modify the elastic LTB capacity, while accounting for these factors.

The elastic critical moment M_{cr} is the capacity obtained from Eq. 1 multiplied by coefficients that take into account the loading and boundary conditions. The moment modification factors are typically functions of bending moments or compression flange stresses at different points along the unbraced length of the beam.

Existing formulations in literature are obtained numerically, and each of them is only accurate within a specific range of use as discussed in the course of this paper. This is established using a general analytical framework that provides more accurate expressions for the LTB moment modification factor. The current study focuses on the evaluation of critical moments using the energy method for two types of boundary conditions (fully fixed and simply-supported) subjected to different loading conditions, such as linear moment gradients, a concentrated load at midspan, and uniformly distributed loads. Given that the closed-form solution to the differential equation of equilibrium for the different moment-gradient cases are unknown, the Rayleigh-Ritz approach is used here to derive these solutions. The deflection functions (i.e., the functions for the twist of the cross-section and lateral deflections) are assumed, such that the boundary conditions for all considered cases are satisfied. The LTB strength is then obtained by minimizing the total potential energy of the system. The energy method has been used by many (Galambos and Surovek 2008; Timoshenko and Gere 1961; Yoo and Lee 2012) to arrive at the elastic critical buckling load for simply-supported beams subjected to concentrated loads and uniformly distributed load, and a cantilever beam subjected to concentrated load.

The results obtained from the Rayleigh-Ritz method are validated with finite element simulations, and also compared against different solutions in the literature. This work seeks to establish accurate formulations for the prediction of the elastic critical moment of steel beams for several common cases of loading and end-restraint conditions, while also evaluating the conditions for which the existing equations are most suitable.

2. Evaluation of M_{cr} using existing design codes and empirical formulae

The basic critical moment equation proposed by Timoshenko and Gere (1916) in Eq. 1 has been modified by others for different loading and boundary conditions using empirical equations. Some of these are described in this section.

1.1 American Institute of Steel Construction (AISC 2016)

The nominal elastic lateral–torsional buckling moment of doubly-symmetric I-sections as per AISC (2016) is calculated using Eq. 2. The flexural strength equation given in AISC (2016) yields identical results to that specified in Eq. 1.

$$M_{cr} = S_x \frac{C_b \pi^2 E}{\left(\frac{L}{r_{ts}}\right)^2} \sqrt{1 + 0.078 \frac{J}{S_x h_o} \left(\frac{L_b}{r_{ts}}\right)^2} \tag{2}$$

where, $r_{ts}^2 = \frac{\sqrt{I_y I_w}}{S_x}$, S_x is the elastic section modulus taken about the major axis, C_b is the lateral-torsional buckling modification factor and h_o is the distance between the flange centroids.

The moment modification factor, C_b used in AISC (1961, 1986) is given by Eq. 3 which gives a lower bound of the solution proposed by Salvadori (1956). However, this equation is applicable primarily for linear moment gradient loading cases.

$$C_b = 1.75 + 1.05\left(\frac{M_1}{M_2}\right) + 0.3\left(\frac{M_1}{M_2}\right)^2 \leq 2.3 \tag{3}$$

where; M_1 is the smaller moment at the end of the unbraced length, and M_2 is the larger moment at the end of the unbraced length. The later editions of the AISC specification (AISC 2016) provide the moment modification factor based on Kirby and Nethercot (1979), given in Eq. 4 which is also valid for nonlinear moment gradient loading.

$$C_b = \frac{12.5 M_{\max}}{2.5 M_{\max} + 3 M_A + 4 M_B + 3 M_C} \leq 2.3 \tag{4}$$

where; $M_{\max}$ is the absolute value of the maximum moment in the unbraced segment, and M_A, M_B, M_C are the absolute values of the bending moments at the quarter-, mid-, and three-quarter points of the unbraced segments, respectively.

Eqs. 3 and 4 are applicable for warping-free conditions, the use of these equations for warping-fixed conditions can lead to unconservative estimates of the M_{cr} as shown in this paper.

1.2 British Standard, BS 5950

BS 5950-1 (2000), makes use of an equivalent uniform moment factor m_{LT} to take into account the moment gradient effects. The inverse of m_{LT} is the same as the moment modification factor, C_b, given in Eq. 5. It can be observed that the equation is similar to Eq. 4, however, slightly different coefficients are considered for reverse curvature bending in simply-supported beams.

$$C_b = \frac{M_{\max}}{0.2 M_{\max} + 0.15 M_A + 0.5 M_B + 0.15 M_C} \leq 2.273 \tag{5}$$

Eurocode (CEN, 2005) does not provide specific equations to calculate M_{cr} or C_b, but Gardner and Nethercot (2005) propose modifications similar to those in BS 5950-1 (2000). Hence, Eurocode is not considered separately in the current studies.

1.3 Contemporary literature

While using the equations proposed by (Kirby and Nethercot 1979; Salvadori 1956) one can predict the elastic LTB strength that agrees well with the actual strength within a limited error for simply-supported boundary conditions. The results are shown to be unconservative for warping-fixed conditions. Nethercot and Rockey (1972) provided solutions that can be used for the warping-fixed condition, for a concentrated load at midspan and a uniformly distributed load, both applied at the shear center in Eqs. 6 and 7.

$$C_b = 1.916 - 0.426\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right)^2 + 1.851\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right) \tag{6}$$

$$C_b = 2.093 - 0.947\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right)^2 + 3.117\left(\frac{\pi}{L}\sqrt{\frac{EI_w}{GJ}}\right) \tag{7}$$

Serna et al. (2006), using finite difference and finite element methods, proposed a general expression for the equivalent uniform moment factor. The finite element analyses were conducted for two cross-sections, and two different unbraced lengths. The authors studied laterally and flexurally simply-supported (i.e., the elastic effective length factor, K=1), and fixed conditions. (i.e., the effective length factor, K=0.5). The authors also looked at both warping restraint and warping-free conditions (i.e., the warping restraint factor, K_w=1, and K_w=0.5). They observed that AISC (1994) was conservative for simply-supported beams and unconservative for beams that were fixed both laterally and torsionally. The solution provided by them is presented in Eq. 8 and is valid for doubly-symmetric steel I-beams transversely loaded at the shear center for any given support condition and moment distribution along the unbraced length.

$$C_b = \frac{\sqrt{\sqrt{k}A_1 + \left[\frac{\left(1-\sqrt{k}\right)}{2}A_2\right]^2} + \frac{\left(1-\sqrt{k}\right)}{2}A_2}{A_1} \tag{8}$$

where; $A_1 = \frac{M_{\max}^2 + 9kM_A^2 + 16M_B^2 + 9kM_C^2}{\left[1+9k+16+9k\right]M_{\max}^2}, A_2 = \left|\frac{M_{\max} + 4M_1 + 8M_A + 12M_B + 8M_C + 4M_2}{37M_{\max}}\right|,$

k is equal to 1 if both lateral bending and warping are free, and equal to 0.5 if both lateral bending and warping are prevented at the ends of the unbraced segment.

Wong and Driver (2010) compared the equivalent moment factors from numerical results with those described in various literature, for a wide range of moment loading conditions. They found that all available methods predict the moment capacities well for linear moment distributions in single curvature, and reverse curvature bending with a k (ratio of end moments) value up to 0.5. The results from AISC (2016) were found to be conservative up to 18% for reverse curvature bending. Although the quarter-point moment equation proposed by Serna et al. (2006) was able to capture the trends of the numerical data, the results however seem to exceed the numerical solutions for a beam subjected to a uniformly distributed load with one end moment. An improved method was proposed by Wong and Driver using quarter-point moments given in Eq. 9, which applies only to warping-free conditions.

$$C_b = \frac{4M_{\max}}{\sqrt{M_{\max}^2 + 4M_A^2 + 7M_B^2 + 4M_C^2}} \le 2.5 \tag{9}$$

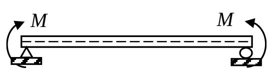
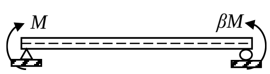
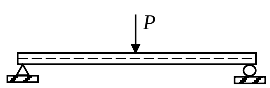
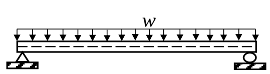
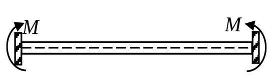
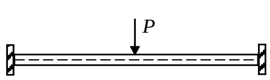
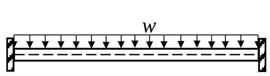
These modification factors, which are used along with the elastic LTB solution presented by Timoshenko and Gere (1916), are usually fit to finite element or finite difference methods for moment gradients. These equations are derived for specific loading conditions, but are universally applied to all loading and boundary conditions. Most of these formulations are accurate within a limited range of applicability. Therefore, in this work, a general theoretical model based on the Rayleigh-Ritz formulation is proposed where closed-form solutions for a wide range of boundary and loading conditions are presented.

3. LTB Strength Using Rayleigh-Ritz Formulation

This section provides the elastic lateral-torsional buckling solutions obtained using the Rayleigh-Ritz method for both simply-supported and fixed beams, subjected to various loading conditions as listed in Table 1.

The critical buckling moment solutions are found to be a factor of M_{ocr} derived by Timoshenko for the basic case of uniform moment and fork boundary conditions. Thus, these solutions can also be used to directly calculate the LTB moment modification factor, C_b, as the ratio of M_{cr} with M_{ocr}. Similarly, the fixed beams are subjected to a uniform moment, concentrated load at the midspan, and a uniformly distributed load. These solutions are obtained by using the principle of virtual work.

Table 1: Loading and boundary conditions studied using the energy method

Case study No.	Bending moment diagram	In-plane flexural boundary condition	Out-of-plane flexural boundary condition	Warping
1		simply-supported	simply-supported	free
2		simply-supported	simply-supported	free
3		simply-supported	simply-supported	free
4		simply-supported	simply-supported	free
5		fixed	fixed	fixed
6		fixed	fixed	fixed
7		fixed	fixed	fixed

The following assumptions are made to derive the LTB equation for the I-beams:

1. The I-beams are subjected to bending about their major principal axis.
2. The beams are doubly symmetric.
3. The assumed deflections and twists are small.
4. The material is linear-elastic, homogenous, and isotropic.

5. The flanges and webs are compact, to preclude local buckling.
6. No distortion occurs in the cross-section during buckling.

The base case of a doubly-symmetric I-beam with fork boundary conditions subjected to uniform moments is shown in Figure 1. In this figure, x and y are the distances measured along the major and minor axes of the cross-section, z is the distance along the span of the beam, L is the unbraced length of the member, ϕ is the twist of the cross-section, and v and u are the in-plane (y-axis) and lateral/ out-of-plane (x-axis) deflections, respectively. The result obtained for this reference case using the energy formulation is compared with Timoshenko's solution.

In the principle of virtual work, the total potential, Π is determined as the sum of the elastic energy of the system, U and the potential of the external forces, V. The total potential is a constant.

$$\Pi = U + V \tag{10}$$

The total elastic strain energy of the beam is given by

$$U = \frac{1}{2}\int_0^L \left\{ EI_y \left(u''\right)^2 + EI_w \left(\phi''\right)^2 + GJ\left(\phi'\right)^2 \right\} dz \tag{10a}$$

And the potential of external force is equal to

$$V = \frac{1}{2}\int_0^L M_x \left\{ 2\phi\left(u''\right) \right\} dz \tag{10b}$$

The total potential of the system can hence be written as

$$\Pi = \frac{1}{2}\int_0^L \left\{ EI_y \left(u''\right)^2 + EI_w \left(\phi''\right)^2 + GJ\left(\phi'\right)^2 \right\} dz + \frac{1}{2}\int_0^L M_x \left\{ 2\phi\left(u''\right) \right\} dz \tag{10c}$$

where; M_x is the moment about the major axis, u'' is the second derivative of the lateral deflection u, ϕ' is the first derivative of the twist ϕ, and ϕ'' is the second derivative of ϕ.

It is important to note that the potential energy formulation is a function of loading conditions or the bending moment (M), and unknown coefficients of the displacement functions (u and ϕ). It is shown in Section 1.11 that the assumed number of terms in the displacement functions provides sufficient accuracy. These unknowns may be calculated by applying the Rayleigh-Ritz technique in Eq. 10c, minimizing the potential energy of the system ($\delta\Pi = 0$).

This method is applied to different boundary and loading conditions, a few of which are discussed in the following subsections.

1.4 Simply-supported beam subjected to uniform moment

A simply-supported beam subjected to uniform moment is shown in Figure 1. Here, the solution from the Rayleigh-Ritz method is shown to be identical to the solution provided by Timoshenko and Gere (1916).

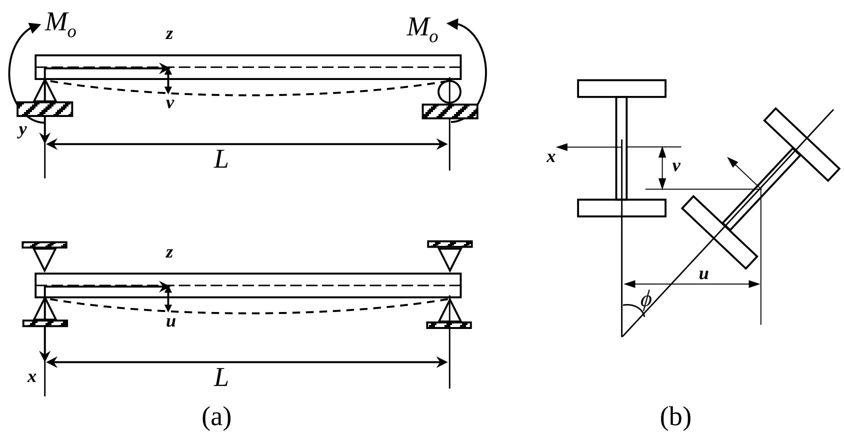


Figure 1: (a) longitudinal profile and (b) cross-section of the simply-supported beam

The boundary conditions for a simply-supported beam are given by

$$u = u'' = \phi = \phi'' = 0 \qquad \text{at } z = 0 \text{ and } z = L \tag{11}$$

The assumed displacement functions are

$$u = A\sin\frac{\pi z}{L}, \phi = B\sin\frac{\pi z}{L} \tag{12}$$

The bending moment at any section along the length of the beam, $M_x = M$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = U + V = \frac{\pi^4 EI_y A^2}{4L^3} + \frac{\pi^4 EI_w B^2}{4L^3} + \frac{\pi^2 GJB^2}{4L} - \frac{\pi^2 MAB}{2L} \qquad 13)$$

Differentiating the total energy with respect to the unknowns *A* and *B*, the following Eqs. 14a and 14b are obtained.

$$\frac{\partial \Pi}{\partial A} = \frac{\pi^4 EI_y A}{2L^3} - \frac{\pi^2 MB}{2L} \tag{14a}$$

$$\frac{\partial \Pi}{\partial B} = \frac{\pi^4 EI_w B}{2L^3} + \frac{\pi^2 GJB}{2L} - \frac{\pi^2 MA}{2L} \tag{14b}$$

The total potential being constant, Eqs. 14a and 14b are equated to zero.

$$\begin{bmatrix} \frac{\pi^2 EI_y}{L^2} & -M \\ -M & \frac{\pi^2 EI_w}{L^2} + GJ \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = 0$$

Thereby, the solution for critical elastic lateral-torsional moment (M_{ocr}) for the base case can be obtained by evaluating the determinant of the above matrix.

$$M_{ocr} = \sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{15}$$

Eq. 15 is the same as the equation derived by Timoshenko and Gere (1916).

1.5 Simply-supported beam subjected to varying moment

A simply-supported beam subjected to a linearly varying moment is shown in Figure 2. The factor β, which is the ratio of end moments, is positive for single curvature, and negative for reverse curvature bending.

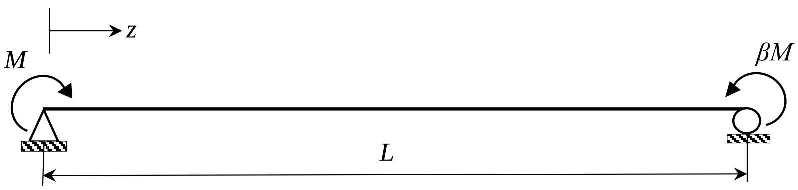


Figure 2: Simply-supported beam subjected to linearly varying moments.

Based on the boundary conditions, the assumed displacement functions are given as follows:

$$u = A\sin\frac{\pi z}{L} + B\sin\frac{2\pi z}{L}, \phi = C\sin\frac{\pi z}{L} \tag{16}$$

The bending moment at a distance z from the left support is given by $M_x = M\left\{1-(1-\beta)\frac{z}{L}\right\}$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{\pi^4 EI_y\left(A^2+16B^2\right)}{4L^3} + \frac{\pi^4 EI_w C^2}{4L^3} + \frac{\pi^2 GJC^2}{4L} + \frac{MC}{36L}\left[128B(\beta-1) - 9\pi^2 A(\beta+1)\right] \tag{17}$$

The total potential being constant, the differentiation of the total energy with respect to the unknowns is equated to zero. The critical elastic LTB moment can be obtained by solving the simultaneous equations as given in Eq. 18.

$$M_{cr} = \frac{1}{\sqrt{\left\{0.5(1+\beta)\right\}^2 + \left\{0.18(1-\beta)\right\}^2}}\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{18}$$

Eq. 18 can be expressed as $C_b M_{ocr}$, where $C_b = \frac{1}{\sqrt{\left\{0.5(1+\beta)\right\}^2 + \left\{0.18(1-\beta)\right\}^2}}$. This equation is later compared with empirical solutions and finite element solutions for a range of values of β for both single curvature and reverse curvature.

1.6 Simply-supported beam subjected to a concentrated load at the mid-span

A simply-supported beam subjected to a concentrated load at midspan is shown in Figure 3. The assumed displacement functions are the same as those in Eq. 16.

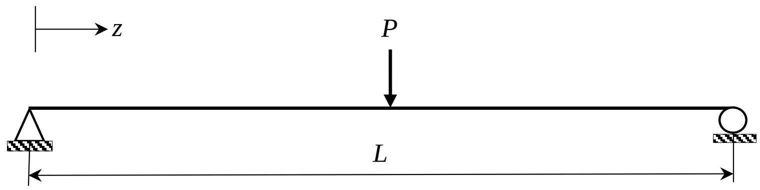


Figure 3: Simply-supported beam with a concentrated load at the center

The bending moment at a distance *z* from the left support is given by

$$M_x = \frac{Pz}{2} \qquad \text{for } z \le \frac{L}{2}$$
$$= \frac{P(L-z)}{2} \qquad \text{for } z \ge \frac{L}{2}$$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{\pi^4 EI_y\left(A^2+16B^2\right)}{4L^3} + \frac{\pi^4 EI_w C^2}{4L^3} + \frac{\pi^2 GJC^2}{4L} + \frac{4(4-3\pi)PBC}{18} - \frac{\left(4+\pi^2\right)PAC}{32} + \frac{4(4+3\pi)PBC}{18} - \frac{\left(-4+3\pi^2\right)PAC}{32} \tag{19}$$

Considering the total potential to be constant, the differentiation of total energy with respect to the unknowns is equated to zero. The critical elastic buckling load can be obtained by solving the simultaneous equations, and is given by Eq. 20.

$$P_{cr} = \frac{8\pi^2}{L\left(\pi^2+4\right)}\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2}+GJ\right)} \tag{20}$$

The critical elastic LTB moment can be obtained as given in Eq. 21 by recognizing that $M_{cr} = \frac{P_{cr}L}{4}$,

$$M_{cr} = 1.423\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2}+GJ\right)} \tag{21}$$

Eq. 21 can be expressed as $C_b M_{ocr}$, where, C_b is equal to 1.423.

1.7 Simply-supported beam subjected to a uniformly distributed load

A simply-supported beam subjected to a uniformly distributed load is shown in Figure 4. The assumed displacement functions are the same as those given by Eq. 16.

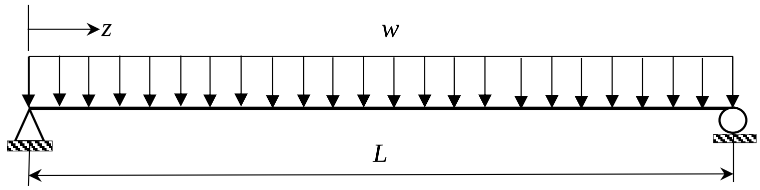


Figure 4: Simply-supported beam with uniformly distributed loads

The bending moment at a cross-section at a distance z from the left support is given by $M_x = \frac{w}{2}\left(Lz - z^2\right)$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{\pi^4 EI_y A^2}{4L^3} + \frac{4\pi^4 EI_y B^2}{L^3} + \frac{\pi^2 GJC^2}{4L} + \frac{\pi^4 EI_w C^2}{4L^3} - \frac{\left(3+\pi^2\right) ACLw}{24} \tag{22}$$

Considering the total potential to be constant, the differentiation of the total energy with respect to the unknowns can be equated to zero. Solving the system of simultaneous equations, the critical elastic buckling load is obtained as

$$w_{cr} = \frac{12\pi^2}{L^2\left(\pi^2+3\right)}\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{23}$$

The critical elastic LTB moment can be obtained as given in Eq. 24, by using the relation, $M_{cr} = \frac{w_{cr}L^2}{8}$

$$M_{cr} = 1.1503\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{24}$$

Again, M_{cr} can be expressed as $C_b M_{ocr}$, where, C_b is equal to 1.15.

1.8 Fixed-fixed beam subjected to uniform moment

A fixed beam subjected to uniform moment is shown in Figure 5. This beam is fixed both in-plane and out-of-plane at its supports. The boundary condition for in-plane fixity is only used to determine the relationship between the buckling load and the buckling moment. However, for the discussion on the beam subjected to uniform moments in this section, the solution would be the same even if the beam were simply-supported in-plane, and fixed out-of-plane.

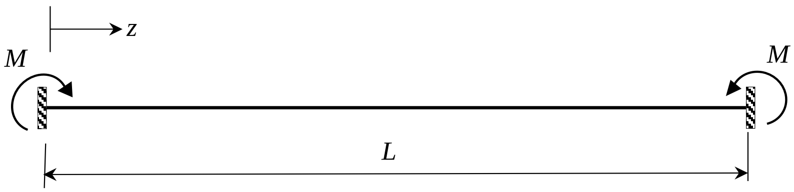


Figure 5: Fixed-fixed beam with uniform moments

The boundary conditions for a beam with lateral bending and torsion fixed are given by

$$u = u' = \phi = \phi' = 0 \qquad \text{at } z = 0 \text{ and } z = L \tag{25}$$

The assumed displacement functions are

$$u = A\left(1 - \cos\frac{2\pi z}{L}\right), \phi = B\left(1 - \cos\frac{2\pi z}{L}\right) \tag{26}$$

The bending moment at any distance *z* is constant, and $M_x = M$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{4\pi^4 EI_y A^2}{L^3} + \frac{4\pi^4 EI_w B^2}{L^3} + \frac{\pi^2 GJB^2}{L} - \frac{2\pi^2 MAB}{L} \tag{27}$$

The total potential being constant, differentiation of the total energy with respect to the unknowns can be equated to zero. The critical elastic LTB moment can be obtained by solving the system of simultaneous equations as given in Eq. 28.

Eq. 28 indicates that the effective length factor, *K* for the fixed-ended beam is 0.5, i.e., on substituting 0.5*L* in place of *L* in the equation for M_{ocr} (Eq. 15), one may obtain Eq. 28 as the base $M_{ocr\text{-}f}$ for the fixed condition. Eq. 28 can also be used for beams that are simply-supported in the in-plane direction, and fixed only in the out-of-plane direction (warping and torsion are fixed).

$$M_{ocr-f} = \sqrt{\frac{\pi^2 EI_y}{\left(\frac{L}{2}\right)^2}\left(\frac{\pi^2 EI_w}{\left(\frac{L}{2}\right)^2} + GJ\right)} \tag{28}$$

1.9 *Fixed-fixed beam subjected to concentrated load applied at the shear center at midspan*

A fixed beam subjected to a concentrated load at midspan is shown in Figure 6. The displacement boundary conditions are given in Eq. 25.

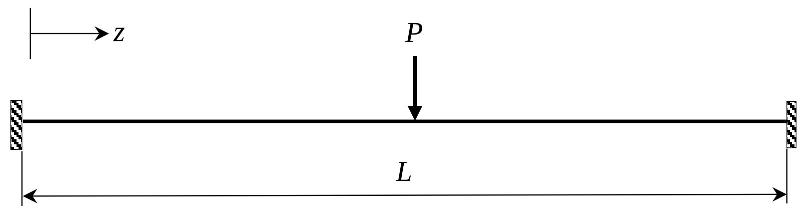


Figure 6: Fixed-fixed beam with a concentrated load at the center

The assumed displacement functions are

$$u = A\left(1-\cos\frac{2\pi z}{L}\right) + B\left(1-\cos\frac{4\pi z}{L}\right), \phi = C\left(1-\cos\frac{2\pi z}{L}\right) \tag{29}$$

The bending moment at a distance *z* is given by

$$M_x = \frac{Pz}{2} - \frac{PL}{8} \qquad \text{for } z \le \frac{L}{2}$$

$$= \frac{P(L-z)}{2} - \frac{PL}{8} \qquad \text{for } z \ge \frac{L}{2}$$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{4\pi^4 EI_y A^2}{L^3} + \frac{64\pi^4 EI_y B^2}{L^3} + \frac{40BCP}{9} + \frac{\pi^2 GJC^2}{L} + \frac{4\pi^4 EI_w C^2}{L^3} - 2ACP \tag{30}$$

As before, solving a set of simultaneous equations, considering the total potential to be constant, the critical elastic buckling load is obtained as

$$P_{cr} = \left(\frac{9\pi^2}{\sqrt{106}L}\right)\sqrt{\frac{\pi^2 EI_y}{\left(\frac{L}{2}\right)^2}\left(\frac{\pi^2 EI_w}{\left(\frac{L}{2}\right)^2} + GJ\right)} \tag{31}$$

The critical elastic LTB moment can be obtained from $M_{cr} = \frac{P_{cr}L}{8}$

$$M_{cr} = 1.0784\sqrt{\frac{\pi^2 EI_y}{\left(\frac{L}{2}\right)^2}\left(\frac{\pi^2 EI_w}{\left(\frac{L}{2}\right)^2} + GJ\right)} \tag{32}$$

Eq. 32 can be obtained by using C_b equal to 1.078, when compared against the reference case of a fixed-fixed beam subjected to a uniform moment in the equation for $M_{ocr\text{-}f}$ (Eq. 28).

1.10 Fixed beam subjected to uniformly distributed load

The schematic representation of a fixed beam subjected to uniform moment is shown in Figure 7. The displacement boundary conditions are given in Eq. 25 and the assumed displacement functions are the same as those in Eq. 29.

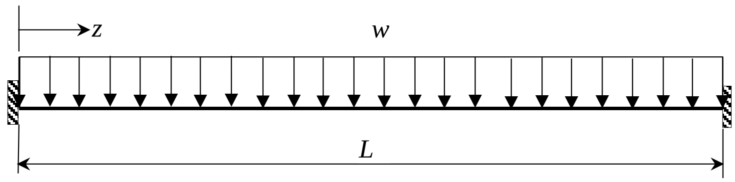


Figure 7: Fixed-fixed beam with uniformly distributed loads

The bending moment at a distance z is given by, $M_x = \frac{w}{12}\left(6Lz - L^2 - 6z^2\right)$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{4\pi^4 EI_y A^2}{L^3} + \frac{64\pi^4 EI_y B^2}{L^3} + \frac{\pi^2 GJC^2}{L} + \frac{4\pi^4 EI_w C^2}{L^3} - \frac{7LCAw}{8} + \frac{11LCBw}{9} \tag{33}$$

Considering the total potential to be constant, and solving the simultaneous equations, the critical elastic buckling load is obtained as

$$w_{cr} = \left(\frac{144\pi^2}{\sqrt{4453}L^2}\right)\sqrt{\frac{\pi^2 EI_y}{\left(\frac{L}{2}\right)^2}\left(\frac{\pi^2 EI_w}{\left(\frac{L}{2}\right)^2} + GJ\right)} \tag{34}$$

The critical elastic LTB moment can be obtained from the relation $M_{cr} = \frac{w_{cr}L^2}{12}$ as given in Eq. 35.

$$M_{cr} = 1.7748\sqrt{\frac{\pi^2 EI_y}{\left(\frac{L}{2}\right)^2}\left(\frac{\pi^2 EI_w}{\left(\frac{L}{2}\right)^2} + GJ\right)} \tag{35}$$

Here C_b equals to 1.77, when compared against Eq. 28.

1.11 Convergence study

This section discusses the number of polynomials required in the assumed displacement functions. The convergence of the solution is demonstrated by considering a larger number of polynomials in the assumed shape functions for all the cases studied in Sections 1.4-1.10. In this section, only the solution for simply-supported beams subjected to a concentrated load at midspan is presented. The results from the studies show that the obtained solutions in Sections 1.4-1.10 are within 6 % of the exact value.

The assumed displacements for a simply-supported are given by

$$u = A\sin\frac{\pi z}{L} + B\sin\frac{2\pi z}{L} + C\sin\frac{4\pi z}{L}, \phi = D\sin\frac{\pi z}{L} \tag{36}$$

The bending moment for the beam subjected to a concentrated load at midspan is given by

$$\begin{aligned} M_x &= \frac{Pz}{2} && \text{for } z \le \frac{L}{2} \\ &= \frac{P(L-z)}{2} && \text{for } z \ge \frac{L}{2} \end{aligned}$$

The total potential energy calculated using Eq. 10c is given by

$$\Pi = \frac{\pi^4 EI_y\left(A^2 + 16B^2 + 256C^2\right)}{4L^3} + \frac{\pi^2 D^2\left(GJL^2 + \pi^2 EI_w\right)}{4L^3} - \frac{\left(4+\pi^2\right)ADP}{16} \tag{37}$$

Considering the total potential to be minimum, the differentials of the total potential with respect to unknowns can be equated to zero. Solving the simultaneous equations, we get the critical elastic buckling load as

$$P_{cr} = \frac{8\pi^2}{L\left(\pi^2+4\right)}\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{38}$$

The critical elastic LTB moment can be obtained from $M_{cr} = \frac{P_{cr}L}{4}$ as given in Eq. 39.

$$M_{cr} = 1.4232\sqrt{\frac{\pi^2 EI_y}{L^2}\left(\frac{\pi^2 EI_w}{L^2} + GJ\right)} \tag{39}$$

The M_{cr} given in Eq. 39 is the same as the critical elastic LTB moment obtained in Eq. 21. This shows that number of terms considered in the assumed solution are sufficient. Repeating this exercise for other loading and boundary conditions discussed in this paper shows that the results are within 6% of the accurate value.

4. Numerical Simulations Using Finite Element Analyses

The elastic buckling analyses are performed using SABRE2 (White et al. 2018), a structural analysis and design software to compare the critical buckling moment obtained in Sections 1.4-1.10. SABRE2 (White et al. 2018) employs 7-degree of freedom beam elements including the warping degree of freedom. The solutions from SABRE2 are also verified with a few studies using buckling analyses in (ABAQUS 2022). Following a mesh convergence study, eight elements are used per unbraced segment in the simulations. The boundary conditions used to simulate the simply-supported beams and fixed beams are shown in Figure 8.

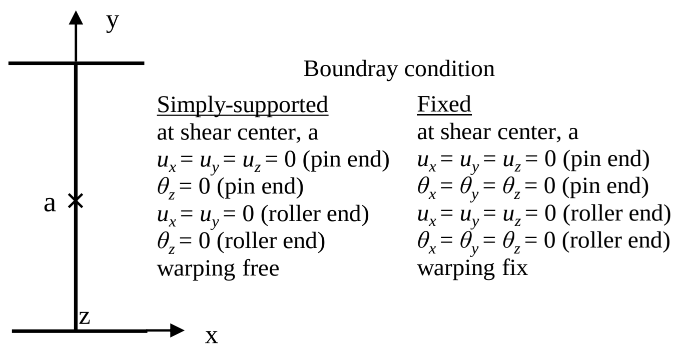


Figure 8: Boundary conditions for simply-supported and fixed beams

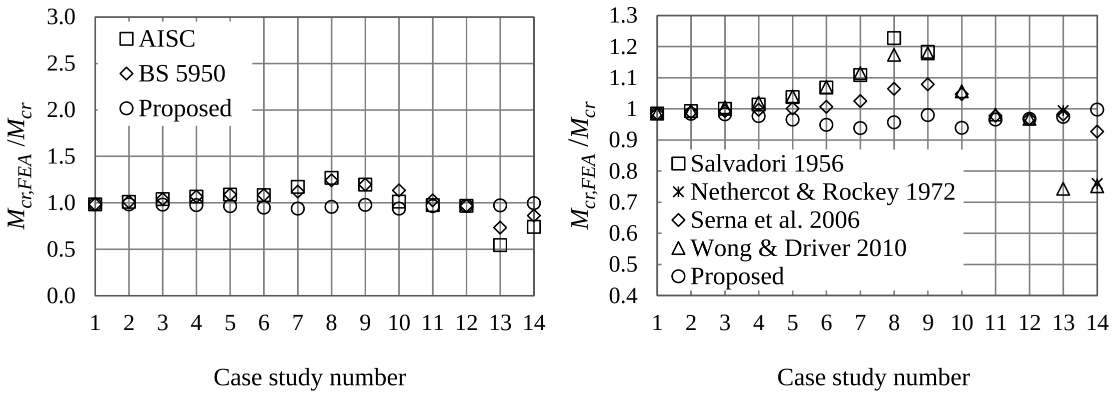
5. Comparative Study and Discussions

Two narrow parallel flange (NPB) sections are chosen from BIS (2004) for the validation studies. The dimensions of the chosen sections are shown in Table 2. The chosen sections are classified as plastic sections (Class I) as per BIS (2007). The critical elastic LTB moment calculated using the formulae developed in Sections 1.4-1.10, $M_{cr,energy}$, and the elastic LTB capacity obtained from the finite element analysis (FEA), $M_{cr,FEA}$ for the chosen sections for the 14 different cases are compared in Table 3.

Cases 1-11, shown in Table 3 are flexurally and torsionally simply-supported, while cases 12-14 are modelled as flexurally-fixed and warping-fixed boundary conditions. The results obtained are also compared with the existing solutions given in various National/International codes (AISC 2016; BS 5950-1 2000) and literature (Salvadori 1956; Serna et al. 2006; Wong and Driver 2010) in Figures 9-12. Figures 9(a) and 10(a) compare the empirical equation given in codes (AISC 2016; BS 5950-1 20005) and the proposed equations with the critical elastic LTB moment obtained from FEA for NPB 400, and 600, respectively. Similarly, Figures 9(b) and 10(b) compare the empirical equation given in the literature (Nethercot and Rockey 1972; Salvadori 1956; Serna et al. 2006; Wong and Driver 2010) and the proposed equations with the critical elastic LTB moment obtained from FEA for NPB 400 and 600, respectively. In Figures 9 and 10, the y-axis is the normalized elastic critical LTB moment obtained from FEA with the M_{cr} from different formulations, and the numbers in the *x*-axis represent the case number given in Table 3.

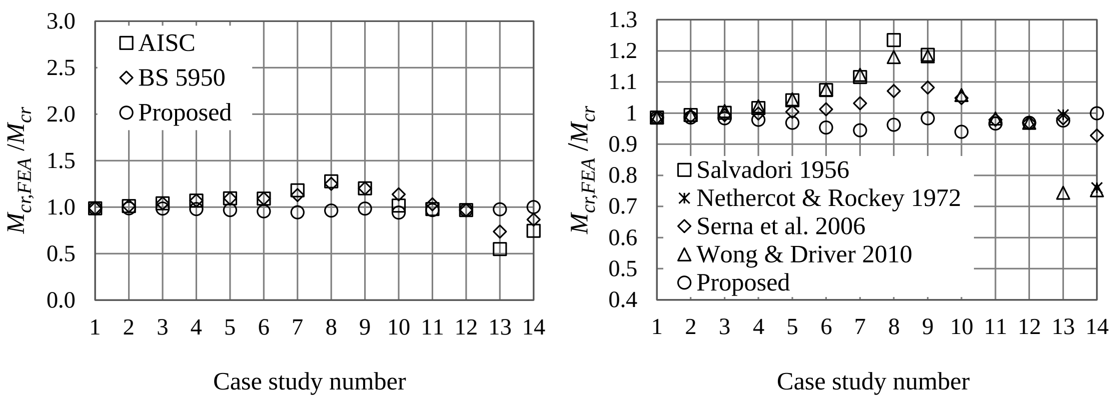
Table 2: Dimensions of the NPB sections (BIS 2004) chosen for the validation studies in mm

Section	Length, L	Web depth, D	Web thickness, t_w	Flange width, b_f	Flange thickness, t_f	D/b_f	$b_f/2t_f$
NPB 400	6000	373	7.0	180	12.0	2.1	7.5
NPB 600	6000	562	9.8	220	17.5	2.6	6.3



(a) (b)

Figure 9: Comparison of M_{cr} obtained from FEA with Rayleigh-Ritz method and equations given in (a) code (b) other empirical methods (NPB 400)



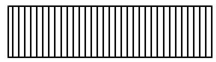
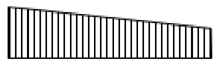
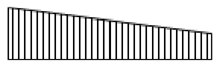
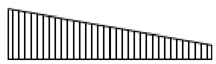
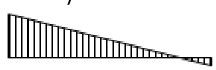
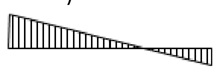
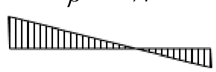
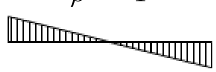
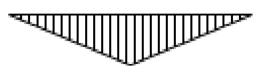
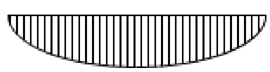
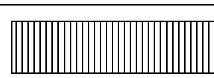
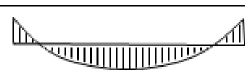
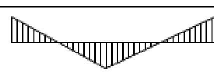
(a) (b)

Figure 10: Comparison of M_{cr} obtained from FEA with Rayleigh-Ritz method and equations given in (a) code (b) other empirical methods (NPB 600)

The following conclusions are drawn from Table 3 and Figures 9 and 10:

1. The calculated elastic LTB moment strength for beams in design codes are typically conservative, and the conservatism increases with an increase in the moment gradient. This is particularly true for the beams subjected to reverse curvature (Cases 6-9). The elastic LTB strength of a simply-supported beam, as calculated using AISC (2016) and British standard (BS 5950-1 2000), are very similar and can be up to 28% and 25% conservative, respectively, for a β value of -0.75.
2. The design code equations tend to be unconservative by up to 45 % for beams with warping fixity. The British standard (BS 5950-1 2000) predicts strengths that are 26% and 14% greater than the true solutions for fixed beams subjected to a concentrated load and a uniformly distributed load, respectively. As previously mentioned and demonstrated in these plots, the use of AISC (2016) equation to determine the elastic LTB strength of beams with the warping-fixed condition is significantly unconservative with an overestimation of up to 45%.

Table 3: Comparison of the elastic critical LTB moment obtained from the energy method with FE solutions

Case study no.	Bending moment diagram	NPB 400	NPB 600
		$M_{cr,energy}/M_{cr,FEA}$	$M_{cr,energy}/M_{cr,FEA}$
Case 1	β = +1 	0.98	0.99
Case 2	β = + ¾ 	0.98	0.98
Case 3	β = + ½ 	0.98	0.98
Case 4	β = + ¼ 	0.98	0.98
Case 5	β = 0 	0.97	0.97
Case 6	β = - ¼ 	0.95	0.95
Case 7	β = - ½ 	0.94	0.94
Case 8	β = - ¾ 	0.96	0.96
Case 9	β = - 1 	0.98	0.98
Case 10		0.94	0.94
Case 11		0.97	0.97
Case 12		0.97	0.97
Case 13		0.97	0.98
Case 14		1.00	1.00

3. The comparison with the literature shows that the equation given by Salvadori (1956), applicable for linear moment gradients estimates the elastic lateral-torsional buckling moment conservatively with a maximum conservatism of 23% for β equal to -0.75. The equation proposed by Wong and Driver (2010) is accurate to conservative up to 18% (for

simply-supported beams with reverse curvature bending), but predicts strengths that are 25% unconservative for Cases 13 and 14 (i.e., beams with warping-fixed condition).

4. The solutions by Nethercot and Rockey (1972), which are also presented in Galambos (1998) for warping-fixed conditions are also unconservative by 25% for Case 14.
5. Serna et al. (2006) predict strengths that are typically lower than the FEA results (up to 8% for simply-supported beams with reverse curvature bending with β equal to -1) and are unconservative (~8%) for a fixed beam subjected to a uniformly distributed load. These equations appear to provide the best estimates of C_b considering different loading and boundary conditions.
6. The LTB capacity estimated using the Rayleigh-Ritz method matches with the numerical results obtained from FEA with a maximum difference of 6% for a simply-supported beam with a concentrated load at midspan.
7. These studies essentially show that the energy formulations would be most beneficial, when applied to simply-supported beams subjected to reverse curvature, and for warping-fixed conditions.

6. Conclusions

While the LTB moment modification equations recommended in design codes and specifications are mostly reasonable, they may either lead to overly conservative or unconservative designs, particularly for reverse curvature loading and warping-fixed conditions. Further, the existing modification factors are empirical in nature. In this study, the closed-form solutions for the elastic LTB capacity of beams with both simply-supported and fixed boundary conditions, and subjected to different moment gradient scenarios are derived using the Rayleigh-Ritz method. The key findings are summarized below:

1. The recommended closed-form solutions from the Rayleigh-Ritz method are compared with the results from FE simulations for specific beam parameters. The proposed equations are applicable with a maximum difference of 6% between the analytical and numerical results.

2. Also, the proposed Rayleigh-Ritz formulation provides a quick and effective means to predict LTB behaviour, where only geometrical and material parameters are required to implement the formulation. The equations provided are also of a simple form and can be easily used by design engineers.

3. The modification factor, C_b for fixed beams should be defined with respect to a fixed beam subjected to uniform moment, rather than the simply-supported beam in Timoshenko's solution.

4. The equations in design codes tend to be conservative for simply-supported beams subjected to reverse curvature, and unconservative for beams with warping-fixed conditions.

5. The equations by Serna et al. appear to be the best solutions considering different loading and boundary conditions.

Although the paper only presents the solutions for a few select cases, the same displacement shape functions may be used for other loading conditions to obtain the corresponding expression for the elastic critical buckling moment. This offers a more rigorous approach to formulating C_b as compared to empirical fits to numerical data.

7. Nomenclature

C_b	Moment modification factor
D	Clear depth of the web
E	Young's modulus of elasticity
G	Modulus of rigidity
I_y	Minor axis moment of inertia of the cross-section
I_w	Warping constant
J	St.-Venant torsion constant
K	Effective length factor
K_w	Warping restraint factor
L	Unbraced length of the selected I-beam
M	Bending moment
M_A, M_B, M_C	Absolute values of the bending moments at the quarter-, mid- and three-quarter points of the unbraced segments, respectively
M_{cr}	Critical elastic lateral-torsional buckling moment
$M_{\max}$	Absolute value of maximum moment in the unbraced segment
M_{ocr}	Critical elastic lateral-torsional moment for simply-supported beam subjected to uniform moment
$M_{ocr\text{-}f}$	Critical elastic lateral-torsional moment for fixed beam subjected to uniform moment
M_1	Smaller moment at the end of unbraced length
M_2	Larger moment at the end of unbraced length
P	Magnitude of the transverse concentrated load
P_{cr}	Elastic critical buckling load
S_x	Elastic section modulus taken about the major axis
U	Elastic strain energy of the system
V	Potential of the external forces
b_f	Width of flange
h_o	Distance between the flange centroids
t_f	Flange thickness
t_w	Web thickness
u	Lateral deflection
v	Vertical deflection
w	Magnitude of the uniformly distributed load
z	Distance measured along the length
Π	Total potential of the system
Φ	Twist of the cross-section
β	Ratio of end moments, negative for reverse curvature bending
$\bar{\lambda}_{LT}$	Non-dimensional slenderness of beam

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